

OPERADS AND KNOT SPACES

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1. INTRODUCTION

Let E_m denote the space of embeddings of the interval $\mathbb{I} = [-1, 1]$ in the cube \mathbb{I}^m with endpoints and tangent vectors at those endpoints fixed on opposite faces of the cube, equipped with a homotopy through immersions to the unknot – see Definition 5.1. By Proposition 5.17, E_m is homotopy equivalent to $\text{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega\text{Imm}(\mathbb{I}, \mathbb{I}^m)$. In [28], McClure and Smith define a cosimplicial object \mathcal{O}^\bullet associated to an operad with multiplication \mathcal{O} , whose homotopy invariant totalization we denote $\widetilde{\text{Tot}}(\mathcal{O}^\bullet)$ – see Definition 2.17 and Definition 2.5 below. Let \mathcal{K}_m denote the m th Kontsevich operad, introduced in [22], whose entries are compactified configuration spaces and which is weakly equivalent to the little m -disks operad [37] – see Definition 4.1 and Theorem 4.5 below.

Theorem 1.1. *The totalization of the Kontsevich operad $\widetilde{\text{Tot}}(\mathcal{K}_m^\bullet)$ is homotopy equivalent to the inverse limit of the Taylor tower approximations for E_m in the calculus of embeddings. Moreover, $\widetilde{\text{Tot}}^n(\mathcal{K}_m^\bullet)$ is the n th degree approximation.*

Building on work of Goodwillie, Klein and Weiss [46, 16, 20, 19], and Volic [43, 44], we have the following.

Corollary 1.2. *For $m > 3$, E_m is weakly equivalent to $\widetilde{\text{Tot}}(\mathcal{K}_m^\bullet)$. For $m = 3$, all real-valued finite-type invariants of framed knots factor through a map from E_m to $\widetilde{\text{Tot}}(\mathcal{K}_m^\bullet)$.*

Applying the homology spectral sequence of a cosimplicial space, we have the following.

Corollary 1.3. *For $m > 3$, there is a spectral sequence with E^2 page given by the Hochschild cohomology of the degree $m - 1$ Poisson operad and which converges to the homology of E_m .*

These results resolve conjectures of Kontsevich from his address at the AMS Mathematical Challenges Conference in the summer of 2000 [23]. Kontsevich’s insights were motivated by novel combinatorial work of Tourtchine. In [40] Tourtchine gives an algebraic description of the E^1 -term of Vassiliev’s homology spectral sequence closely related to of our Corollary 1.3. Our results are at the level of spaces and show that the disagreement which Tourtchine found between Hochschild cohomology of the Poisson operad and Vassiliev’s E^2 -term is accounted for by the fact that E_m is not the classical knot space but is instead the space of knots with trivialization through immersions.

Our results bring together some recent developments in algebraic topology and its application to fields such as deformation theory and knot theory. In [36] we presented models for spaces of knots, including a cosimplicial model which is analogous to the cosimplicial model for loop spaces. We build on these results in proving Theorem 1.1. In [28] McClure and Smith resolved the integral Deligne conjecture, showing that the totalization of an operad with multiplication has a two-cubes action both in the setting of chain complexes and that of spaces. We apply their results to establish the following.

1991 *Mathematics Subject Classification.* 57Q45, 18D50, 57M27.

Key words and phrases. knot spaces, operads, embedding calculus.

The author is partially supported by NSF grant DMS-0405922.

Theorem 1.4. *For any m , there is a little two-cubes action on $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$. For $m > 3$, E_m is a two-fold loop space.*

We conjecture that this two-cubes action on $\widetilde{\mathrm{Tot}}(\mathcal{K}_m^\bullet)$ is compatible with a two-cubes action on the space of framed knots which has been recently defined by Budney [7], who goes on to show that long knots in dimension three are free over the two-cubes action. In future work we plan to investigate analogues of this freeness result in higher dimensions. A first step will be to construct operations compatible with this two-cubes structure in the homology spectral sequence for an operad with multiplication, as McClure and Smith currently plan to do. On the E^1 -term such operations will presumably coincide with Tourtchine's bracket, defined combinatorially in [40], but through their space-level construction would also be compatible with differentials and extend to further terms.

Some of the technical results developed in this paper may be of independent interest. We fully develop the operad structure, with multiplication, on the simplicial compactification of configurations in Euclidean spaces. An operad structure on the canonical (Axelrod-Singer) compactification is known [14, 25] but does not yield an operad with multiplication. Instead there is a map from Stasheff's A_∞ operad. Our approach to the operad structure on the simplicial variant blends geometry and combinatorics, revealing an operad structure on the standard simplicial model for the two-sphere.

1.1. Acknowledgements. We thank J. McClure and J. Smith for their interest in this project, answers to questions, and especially for writing Section 15 of [29]. We thank M. Markl and J. Stasheff for comments on early versions of this work, and M. Kontsevich for helpful conversations.

CONTENTS

1. Introduction	1
1.1. Acknowledgements	2
2. Background material	2
2.1. Cosimplicial spaces	3
2.2. Operads	5
3. The choose-two operad	7
4. The Kontsevich operad	9
5. Models for spaces of knots and immersions arising from the calculus of embeddings	12
5.1. A brief overview of the calculus of embeddings	12
5.2. Knot space models through homotopy limits of configuration spaces	13
5.3. A closer look at E_m	15
6. The main result	16
7. Observations and consequences	18
7.1. Spectral sequences	18
7.2. A little two-cubes action from the McClure-Smith framework	19
References	20

2. BACKGROUND MATERIAL

Our main results are stated in terms of operads with multiplication and their associated cosimplicial spaces. As a chance to set the choice of definitions and notation which will be most convenient, and as an opportunity to place all standard material together, we review this material here. For a more complete survey we highly recommend [30]. In particular, Section 3 of [30] introduces cosimplicial spaces, and Section 6 briefly introduces operads. The paper [26] gives a more comprehensive introduction to operads.

A reader familiar with these constructions may wish to skip this section and refer back for clarification as needed.

2.1. Cosimplicial spaces.

Definition 2.1. Let Δ denote the category with one object for every non-negative integer and where the morphisms from k to ℓ are the order-preserving maps from $[k] = \{0, \dots, k\}$ to $[\ell] = \{0, \dots, \ell\}$, ordered in the standard way. A cosimplicial object in a category \mathcal{C} is a (covariant) functor from Δ to \mathcal{C} . A simplicial object is a contravariant functor from Δ to \mathcal{C} .

Cosimplicial objects are denoted X^\bullet , where X^k is the image of $[k]$ under the functor, also known as the k th entry. Simplicial objects are denoted X_\bullet . Central in the theory is the standard cosimplicial space Δ^\bullet , whose k th entry is Δ^k , with vertices labelled by $[k]$, and which sends a morphism $[k] \rightarrow [\ell]$ to the linear map which extends this map on vertices.

Every order-preserving map $[k] \rightarrow [\ell]$ can be factored through elementary maps d^i , which are an isomorphism but for one element - namely i - not in their image, and elementary maps s^i , which are an isomorphism but for having i and $i+1$ in $[k]$ both mapping to $i \in [\ell]$. The corresponding maps between entries of simplicial and cosimplicial objects, called (co)face and (co)degeneracy maps, are often taken as a basis for their definition. The definitions are arranged so that the simplices of an ordered simplicial complex form a simplicial set. Indeed, a simplicial set or simplicial space X_\bullet determines a space called its realization and denoted $|X_\bullet|$, defined as the quotient space of the union of $X_i \times \Delta^i$ over all i by the relations $d_j x \times \beta \sim x \times d^j \beta$ and $s_j x \times \beta \sim x \times s^j \beta$ for all $x \in X^i$, $\beta \in \Delta^i$.

Cosimplicial spaces naturally arise when studying mapping spaces. The totalization of a cosimplicial space $\text{Tot} X^\bullet$ is the space of natural transformations from Δ^\bullet to X^\bullet , which is first used tautologically to study mapping spaces as follows.

Definition 2.2. For any $X \in \mathcal{C}$, a category whose categorical product \odot is symmetric monoidal, taking the product of X with itself gives rise to a functor $X^- : \text{FSet}^{op} \rightarrow \mathcal{C}$ which sends S to $\bigodot_{s \in S} X$, where FSet is the category of finite sets. By composing a simplicial set $Y_\bullet : \Delta \rightarrow \text{FSet}^{op}$ with this functor, we obtain a cosimplicial object X^{Y_\bullet} .

Proposition 2.3. *If Y_\bullet is a simplicial set and X is a space, $\text{Tot}(X^{Y_\bullet})$ is homeomorphic to the space of maps from $|Y_\bullet|$ to X .*

For based X and Y_\bullet we may replace X^{Y_\bullet} by the subspace consisting of based maps from Y_n to X . The resulting cosimplicial space, which we denote $X_\star^{Y_\bullet}$, has totalization homeomorphic to the space of based maps from $|Y_\bullet|$ to X . Another interesting example along these lines is the Hochschild simplicial vector space A^{S^1} , whose associated chain complex computes Hochschild homology of a commutative algebra A .

Cosimplicial spaces are intimately connected with homotopy limits (in fact, homotopy limits are defined in terms of cosimplicial spaces in [5]). The nerve of a category C is the simplicial set C_\bullet , with C_i being the collection of i composable morphisms and structure maps defined through composing such maps or inserting identity maps (see for example Chapter 14 of [21]). Denote the realization of the nerve of C by BC , also called the classifying space. Recall that if c is an object of C , the category $C \downarrow c$ has objects which are maps with target c and morphisms given by morphisms in C which commute with these structure maps. The classifying space $B(C \downarrow c)$ is contractible because $C \downarrow c$ has a final object, namely c mapping to itself by the identity morphism. A morphism g from c to d induces a map from $C \downarrow c$ to $C \downarrow d$, so that $B(C \downarrow -)$ is itself a functor from C to spaces.

Definition 2.4. The homotopy limit of a functor E from a small category C to the category of spaces is $\text{Nat}(B(C \downarrow -), E)$, the space of natural transformations from $B(C \downarrow -)$ to E .

For X^\bullet fibrant in the standard model structure on cosimplicial spaces, that is those which satisfy the matching condition 10.4.6 of [5], Theorem 11.4.4 of [5] states $\text{Tot}(X^\bullet) \simeq \varprojlim_\Delta X^\bullet$, an equivalence needed

for many applications. For cosimplicial spaces which do not necessarily satisfy the matching condition, we use an alternate definition of totalization for which this equivalence is a tautology.

Definition 2.5. • Let $\widetilde{\Delta}^\bullet$ be the cosimplicial space whose $[k]$ th entry is $B(\Delta \downarrow [k])$ and whose structure maps are the standard induced maps.

- For a cosimplicial space X^\bullet let $\widetilde{\text{Tot}}X^\bullet$, called the homotopy invariant totalization, denote the space of natural transformations from $\widetilde{\Delta}^\bullet$ to X^\bullet .
- Let $\widetilde{\text{Tot}}^k X^\bullet$ denote the space of natural transformations from the k th coskeleton of $\widetilde{\Delta}^\bullet$ to X^\bullet .
- Let Δ_k denote the full subcategory of Δ whose objects are $[i]$ for $i \leq k$. Let $i_k : \Delta_k \rightarrow \Delta$ be the inclusion functor.

In Section 15 of [29] the notations $\widetilde{\Delta}^\bullet$ and $\widetilde{\text{Tot}}$ are used for any cofibrant replacement for Δ^\bullet and the corresponding totalization in the model structure on cosimplicial spaces where all objects are fibrant (in the usual model structure from [5], all objects are cofibrant). We choose one model of cofibrant replacement for definiteness.

Definition 2.6. Define $\widetilde{\text{Tot}}(X^\bullet)$ to be the homotopy limit of X^\bullet and $\widetilde{\text{Tot}}^k(X^\bullet) \cong \varprojlim (X^\bullet \circ i_k)$.

The cosimplicial category Δ is also intimately related to the category of subsets of a finite set.

Definition 2.7. Let $P(k)$ be the category of all subsets of $[k] = \{0, \dots, k\}$ where morphisms are defined by inclusion. Let $P_0(k)$ be the full subcategory of non-empty subsets.

The connection of this category to the simplicial world is evident in the identification of $BP_0(k)$ with the barycentric subdivision of a k -simplex. We use this identification to define maps $\widetilde{\Delta}^k \rightarrow \Delta^k$, and thus $\text{Tot}X^\bullet \rightarrow \widetilde{\text{Tot}}X^\bullet$ for any X^\bullet , through the identification of $\widetilde{\Delta}^k$ and Δ^k with $B(\Delta \downarrow [k])$ and $BP_0(k)$ respectively. Namely, take the map induced on classifying spaces by the functor which sends some $[n] \xrightarrow{f} [k]$ in $(\Delta \downarrow [k])$ to the image of f , as a subset of $[k]$. There is also a translation between cosimplicial diagrams and those indexed by $P_0(k)$, which we will use in Section 6.

Definition 2.8. Let $c_k : P_0(k) \rightarrow \Delta$ be the functor which sends a subset S to the object in Δ with the same cardinality, and which sends an inclusion $S \subseteq S'$ to the composite $[i] \cong S \subset S' \cong [j]$, where $[i]$ and $[j]$ are isomorphic to S and S' respectively as ordered sets.

The following lemma is a consequence of Theorem 6.4 of [36], along with the observation that $\widetilde{\text{Tot}}^k(X^\bullet) = \varprojlim (i_k \circ X^\bullet)$.

Lemma 2.9. For X^\bullet a cosimplicial space, $\varprojlim (X^\bullet \circ c_k)$ is weakly equivalent to $\widetilde{\text{Tot}}^k X^\bullet$.

For a cosimplicial space there are spectral sequences for the homotopy groups [5] and homology groups [6, 35] of its (homotopy invariant) totalization, which we will apply in Section 7. The homotopy spectral sequence is straightforward, with convergence immediate from its definition through the tower

$$\cdots \leftarrow \text{Tot}^i X^\bullet \leftarrow \text{Tot}^{i+1} X^\bullet \leftarrow \cdots,$$

whose homotopy inverse limit is $\text{Tot}X^\bullet$ and which is a tower of fibrations if X^\bullet is fibrant. Unraveling the definitions we have the following.

Proposition 2.10. For a fibrant cosimplicial space X^\bullet there is a spectral sequence converging to $\pi_*(\text{Tot}X^\bullet)$ with $E_{-p,q}^1 = \bigcap \ker s_*^k \subseteq \pi_q(X^p)$. The d_1 differential is the restriction to this kernel of the map $\Sigma_{i=0}^{p+1} (-1)^i d_*^i : \pi_q(X^{p-1}) \rightarrow \pi_q(X^p)$.

The homology spectral sequence is more subtle in its convergence. It generalizes the Eilenberg-Moore spectral sequence. One of the precise statements as to the convergence of this spectral sequence arising from [6] is as follows.

Theorem 2.11. *For a fibrant cosimplicial space X^\bullet there is a spectral sequence with $E_{-p,q}^1 = \bigcap \ker s^k_* \subseteq H_q(X^p)$. The d_1 differential is the restriction to this kernel of the map*

$$\Sigma_{i=0}^{p+1} (-1)^i d^i_* : H_q(X^{p-1}) \rightarrow H_q(X^p).$$

This spectral sequence converges to $H_(\text{Tot} X^\bullet)$ if X^k is simply connected for all k and $E_{-p,q}^1 = 0$ when $q \geq cp$ for some $c > 1$.*

Alternately, one may arrive at the same spectral sequence from E^2 forward with $E_{-p,q}^1 = H_q(X^p)$ and d_1 defined as before, but not restricted to the kernel of the codegeneracies.

This theorem follows immediately from Theorem 3.2 of [6] and the Universal Coefficient Theorem, as both of Bousfield's conditions, namely that $E_{-p,q}^1 = 0$ if $p > q$ and that only finitely many $E_{-p,q}^1$ with $q - p = n$ are non-zero for any given n , follow from the vanishing with $q \geq cp$ for some $c > 1$.

These spectral sequences apply unchanged to the homotopy invariant totalization, in which case the fibrancy condition can be dropped. If X^\bullet is a cosimplicial space and \underline{X}^\bullet is a fibrant replacement (as given by Proposition 8.1.3 and Theorem 15.3.4 in [21]) then

$$\widetilde{\text{Tot}}(X^\bullet) = \text{Maps}(\widetilde{\Delta}^\bullet, X^\bullet) \simeq \text{Maps}(\widetilde{\Delta}^\bullet, \underline{X}^\bullet) \simeq \text{Maps}(\Delta^\bullet, \underline{X}^\bullet) = \text{Tot}(\underline{X}^\bullet).$$

Because homotopy and homology of the entries and structure maps of \underline{X}^\bullet agree with those of X^\bullet , the identifications of the E^1 -terms of the associated spectral sequences are unchanged.

2.2. Operads. We define non- Σ operads in terms of a well-known [3, 26, 36, 37] category of rooted trees.

Definition 2.12.

- A rooted, planar tree (or rp-tree) is an isotopy class of finite connected acyclic graph with a distinguished vertex called the root, embedded in the upper half plane with the root at the origin so that the vertical coordinate in the plane is a monotone function which increases on each edge as the distance from the root increases. Univalent vertices of an rp-tree (not counting the root, if it is univalent) are called leaves.
- Each edge of the tree is oriented by the direction of the root path, which is the unique shortest path to the root. The vertex of an edge which is further from the root is called its initial vertex, and the vertex closer to the root is called its terminal vertex. We say that one vertex or edge lies over another if the latter is in the root path of the former. A non-root edge is called redundant if its initial vertex is bivalent.
- Given an rp-tree T and a set of edges E the contraction of T by E is the tree T' obtained by, for each edge $e \in E$, identifying its initial vertex with its terminal vertex (altering the embedding of the tree in a neighborhood of e) and removing e from the set of edges.
- Let Υ denote the category of rp-trees, where there is a morphism from T to T' , denoted either $f_{T,T'}$ or c_E , if T' is the contraction of T along the set of non-leaf edges E .
- Both the collection of leaves in an rp-tree and the collections of edges with a given terminal vertex are ordered, using the clockwise orientation of the plane.
- A sub-tree of an rp-tree is a connected sub-graph. A sub-tree is an rp-tree through a linear isotopy which translates v to the origin.

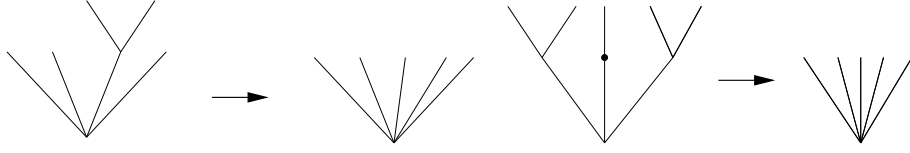
See Figure 2.14 for some examples of objects and morphisms in Υ . Let Υ_n denote the full subcategory of trees with n leaves. Note that Υ_n differs from Φ_n of [36], which is canonically isomorphic to the full subcategory of rp-trees without bivalent vertices. Each Υ_n has a terminal object, namely the unique tree with one vertex, called the n th corolla γ_n as in [26]. We allow for the tree γ_0 which has no leaves, only a root vertex, and is the only element of Υ_0 . For a vertex v let $|v|$ denote the number of edges for which v is terminal, usually called the arity of v .

Definition 2.13. A non- Σ operad is a functor \mathcal{O} from Υ to a symmetric monoidal category (\mathcal{C}, \odot) which satisfies the following axioms.

- (1) $\mathcal{O}(T) = \odot_{v \in T} \mathcal{O}(\gamma_{|v|})$.
- (2) $\mathcal{O}(\gamma_1) = \mathbf{1}_{\mathcal{C}} = \mathcal{O}(\gamma_0)$.
- (3) If e is a redundant edge and v is its terminal vertex then $\mathcal{O}(c_{\{e\}})$ is the identity map on $\odot_{v' \neq v} F(\gamma_{v'})$ tensored with the isomorphism $(\mathbf{1}_{\mathcal{C}} \odot -)$ under the decomposition of axiom (1).
- (4) If S is a subtree of T and if $f_{S,S'}$ and $f_{T,T'}$ contract the same set of edges, then under the decomposition of (1), $F(f_{T,T'}) = F(f_{S,S'}) \odot id$.

We sketch the equivalence of this definition with two standard ones. By axiom (1), the values of \mathcal{O} are determined by its values on the corollas $\mathcal{O}(\gamma_n)$, which corresponds to $\mathcal{O}(n)$ in the usual operad terminology of [27]. Axioms (2) and (3) correspond to the unit condition. By axiom (4), the values of \mathcal{O} on morphisms may be computed by composing morphisms on sub-trees, so we may identify some subset of basic morphisms through which all morphisms factor. In Figure 2.14 we illustrate some basic morphisms in Υ . The first corresponds to what are known as \circ_i operations. The second corresponds May's operad structure maps from Definition 1.1 of [27]. That \mathcal{O} is a functor implies the commutativity of diagrams involving these basic morphisms. Another basic class extending these is that of all morphisms $T \rightarrow \gamma_n$ where γ_n is a corolla and T is any tree.

Figure 2.14.



Two morphisms in Υ which give rise to standard operad structure maps.
The first corresponds to a \circ_i operation, the second to one of May's structure maps.

Example 2.15. • The associative operad \mathcal{A} , defined in any symmetric monoidal category, has $\mathcal{A}(T) = \mathbf{1}_{\mathcal{C}}$, and $\mathcal{A}(T \rightarrow T') = id$ for all morphisms in Υ .
• Let Φ denote the full subcategory of rp-trees with no redundant edges (called the category of reduced trees in [26]) and let $P : \Upsilon \rightarrow \Phi$ denote the functor which contracts all of the redundant edges of an rp-tree. The operad of planar trees, $Tree_n$ from Definition 1.41 of [26], is the operad in the category of sets which sends T to the set of all $T' \in \Phi$ which map to $P(T)$. It sends a contraction of edges of T to the collection of contractions on the corresponding edges for trees over $P(T)$.

Definition 2.16. A map between non- Σ operads is a natural transformation which respects the decomposition of axiom (1) of Definition 2.13. An operad with multiplication is a non- Σ operad \mathcal{O} equipped with a map from the associative operad \mathcal{A} .

The notion of operad with multiplication is due to Gerstenhaber and Voronov [13]. The canonical example is the endomorphism operad of an associative algebra $End(A)$. Algebras over an operad with multiplication are in particular associative algebras. An operad with multiplication in the category of spaces is an operad in the category of pointed spaces.

While we have taken a categorical approach to defining operads, we will take a more coordinatized approach to their associated cosimplicial objects. Recall the \circ_i operations $\circ_i : \mathcal{O}(n) \odot \mathcal{O}(m) \rightarrow \mathcal{O}(n+m-1)$ which provide a basic set of morphisms for an operad, as illustrated in Figure 2.14. From section 3 of [28] we have the following.

Definition 2.17. • Given an operad with multiplication \mathcal{O} , let μ denote the morphism $\mathcal{A}(2) = \mathbf{1}_{\mathcal{C}} \rightarrow \mathcal{O}(2)$.

- Define $d^i : \mathcal{O}(n) \rightarrow \mathcal{O}(n+1)$ by

$$d^i = \begin{cases} \mathbf{1}_{\mathcal{C}} \odot \mathcal{O}(n) \xrightarrow{\mu \odot id} \mathcal{O}(2) \odot \mathcal{O}(n) \xrightarrow{\circ_1} \mathcal{O}(n+1) & \text{if } i = 0 \\ \mathcal{O}(n) \odot \mathbf{1}_{\mathcal{C}} \xrightarrow{id \odot \mu} \mathcal{O}(n) \odot \mathcal{O}(2) \xrightarrow{\circ_i} \mathcal{O}(n+1) & \text{if } 0 < i < n+1 \\ \mathbf{1}_{\mathcal{C}} \odot \mathcal{O}(n) \xrightarrow{\mu \odot id} \mathcal{O}(2) \odot \mathcal{O}(n) \xrightarrow{\circ_2} \mathcal{O}(n+1) & \text{if } i = n+1. \end{cases}$$

- Define s^i as $\mathcal{O}(c_i)$ where $c_i : \gamma_n \rightarrow \gamma_{n-1}$ contracts the i th leaf of γ_n .
- Let \mathcal{O}^\bullet be the cosimplicial object in \mathcal{C} whose n th entry is $\mathcal{O}(n)$ and whose coface and codegeneracy maps are given by d^i and s^i above. If \mathcal{C} is the category of vector spaces over a given field, let $HH^*(\mathcal{O})$ be the homology of the cochain complex defined by the cosimplicial vector space \mathcal{O}^\bullet . If \mathcal{C} is the category of spaces, we call $\widetilde{\text{Tot}}(\mathcal{O}^\bullet)$ the totalization of \mathcal{O}^\bullet .

It is straightforward to show that the maps d^i and s^i satisfy cosimplicial identities.

Remark. In the category of vector spaces, Tourtchine introduced the terminology $HH^*(\mathcal{O})$ because if A denotes an associative algebra and $\text{End}(A)$ is its endomorphism operad then $HH^*(\text{End}(A)) = HH^*(A)$, the usual Hochschild cohomology of A . We are not aware, however, of any sense in which Hochschild cohomology of operads is a cohomology theory for operads. Instead, Kontsevich conjectures that there is a suitable enriched homotopy structure on the category of operads of spaces such that $\widetilde{\text{Tot}}(\mathcal{O}^\bullet)$ is the derived space of maps from the associative operad to \mathcal{O} .

3. THE CHOOSE-TWO OPERAD

At the combinatorial heart of our work is the choose-two operad. Its operad structure intertwines the sets $\binom{\mathbf{n}}{2}$, of distinct pairs of elements $(i, j) \in \mathbf{n} = \{1, \dots, n\}$ with $i < j$ for definiteness, with rooted planar trees. Recall that FSet^{op} , the opposite category to the category of pointed sets, is symmetric monoidal with product given by pointed union, denoted \vee , and unit given by the one-point set. Let S_+ denote the union of a set S with a disjoint base point.

- Definition 3.1.**
- The join of two leaves in an rp-tree is the first vertex (that is, the farthest from the root) at which their root paths coincide.
 - Label both the leaves of an rp-tree and the edges which emanate from a given vertex v with elements of \mathbf{n} and $\{1, \dots, |v|\}$ respectively according to the order given by planar embedding. To an rp-tree T with n leaves and two distinct integers $i, j \in \mathbf{n}$ let v be the join of the leaves labelled i and j and define $J_v(i), J_v(j)$ to be the labels of the edges of v over which leaves i and j lie, as illustrated below.

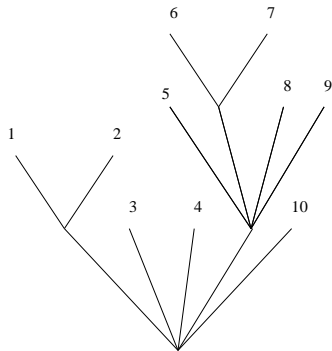


Figure 3.2.

i, j	\longrightarrow	$J_v(i), J_v(j)$
6, 8	\longrightarrow	2, 3
3, 4	\longrightarrow	2, 3
3, 9	\longrightarrow	2, 4
2, 10	\longrightarrow	1, 5

- Let \mathcal{B} , the choose-two operad, be the non- Σ operad in the category \mathbf{FSet}^{op} defined as follows:
 - $\mathcal{B}(T) = \bigvee_w \binom{|w|}{2}_+$, where w ranges over vertices of T .
 - $\mathcal{B}(T \rightarrow \gamma_n)$, where γ_n is a corolla, is the function

$$(i, j) \in \binom{\mathbf{n}}{2} \mapsto (J_v(i), J_v(j)) \in \binom{|v|}{2} \subset \bigvee_{w \in T} \binom{|w|}{2}_+,$$

where v is the join of leaves i and j .

With our choice of definitions, it is straightforward to verify that \mathcal{B} is an operad.

Theorem 3.3. *As a simplicial set, the cosimplicial object in \mathbf{FSet}^{op} associated to the choose-two operad \mathcal{B}^\bullet is isomorphic to S_\bullet^2 , the simplicial model for S^2 .*

Proof. Recall that $S_\bullet^2 = \Delta_\bullet^2 / \partial \Delta_\bullet^2$, where Δ_\bullet^2 is the standard simplicial model for Δ^2 . The set n -simplices of Δ^2 is the set of $(x_0 \leq x_1 \leq \dots \leq x_n) \in \{0, 1, 2\}^{n+1}$, so the cardinality of Δ_n^2 is the $(n+1)$ st triangular number. The i th face and degeneracy maps are defined by deleting and repeating x_i , respectively. To obtain S_\bullet^2 we identify all n -tuples in which one of $\{0, 1, 2\}$ does not appear to a single simplex in each degree, which is degenerate in positive degrees.

The n th entry of S_\bullet^2 is isomorphic to $\binom{\mathbf{n}}{2}_+$, the set of unordered pairs of points in \mathbf{n} , along with a disjoint point $+$ which is the image of $\partial \Delta_\bullet^2$ under the quotient map. The isomorphism records the indices j and k for which $x_{j-1} < x_j$ and $x_{k-1} < x_k$, when there are two such indices. When there are not two such indices, such a sequence is identified with the degenerate point $+$. Under this isomorphism d_i sends $+$ to $+$ and for $i \neq 0, n$ sends

$$(1) \quad (j, k) \mapsto \begin{cases} (\delta_i(j), \delta_i(k)) & \text{if } \delta_i(j) \neq \delta_i(k) \\ + & \text{otherwise} \end{cases} \quad \text{where } \delta_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

For $i = 0$ and $i = n$ the basic formula is the same, but the (j, k) which get sent to $+$ are those with $j = 1$ or $k = n$, respectively. Similarly, s_i sends $+$ to $+$ and sends $(j, k) \mapsto (\sigma_i(j), \sigma_i(k))$ where $\sigma_i(j) = j$ if $j \leq i$ or $j+1$ otherwise.

By definition $\mathcal{B}^n = \binom{\mathbf{n}}{2}_+$, and it is straightforward to check that the structure maps of S_\bullet^2 and the associated cosimplicial object of \mathcal{B}^\bullet coincide. To give an example, we unravel the definition of d^i with

$0 < i < n$ for \mathcal{B}^n . These coface maps are given by composites $(\mathcal{B}(\gamma_n) \vee +) \xrightarrow{id \vee \mu} \mathcal{B}(\bigvee_n^i) \xrightarrow{\circ_i} \mathcal{B}(\gamma_{n+1})$,

where \bigvee_n^i denotes the tree with n root edges and one trivalent internal vertex, which is terminal for the i th root edge. In \mathbf{FSet}^{op} the morphism $id \vee \mu$ corresponds to the collapse map in \mathbf{FSet} which sends $\binom{2}{2}_+ \subset \binom{\mathbf{n}}{2}_+ \vee \binom{2}{2}_+$ to the base point and is the identity on $\binom{\mathbf{n}}{2}_+$. The morphism \circ_i sends $(i, i+1)$ to $(1, 2) \in \binom{2}{2}$ and sends all other (j, k) to $(\delta_i(j), \delta_i(k)) \in \binom{\mathbf{n}}{2}$. The composite of these two maps coincides with the definition of d_i for S_\bullet^2 , as in Equation 1. \square

Let $X^{\mathcal{B}^\bullet}$ be the operad which is the composite of the operad $\mathcal{B}^\bullet : \Upsilon \rightarrow \mathbf{FSet}^{op}$ with the symmetric monoidal functor $X^- : \mathbf{FSet}^{op} \rightarrow \mathbf{Top}$. Theorem 3.3 implies the following.

Corollary 3.4. *For any X in a category \mathcal{C} whose categorical product is symmetric monoidal, $X^{S_\bullet^2}$ canonically defines an operad through its isomorphism with $X^{\mathcal{B}^\bullet}$.*

We have yet to find any familiar interpretation for algebras over this operad in the categories of spaces and vector spaces. For spaces the operad structure on $X^{S_\bullet^2}$ does have consequences, as we explain in Example 7.4.

4. THE KONTSEVICH OPERAD

In this section we define an operad structure on the completion of configurations in Euclidean space up to translation and scaling defined by Kontsevich [22]. The fact that one could define operads using the canonical completion of configuration spaces was noticed by Getzler and Jones [14] soon after this completion was introduced by Fulton-MacPherson [11] and Axelrod-Singer [1]. This operad structure was fully developed by Markl [25]. The variant with which we work was first proposed by Kontsevich [22], but Gaiffi [12] pointed out the deviation with the canonical completion. Indeed, while Kontsevich called the following the Fulton-MacPherson operad, we call it the Kontsevich operad to highlight the difference between the two. Though this construction lacks some of the properties of the canonical completion, in particular smoothness, it has diagonal and projection maps which satisfy simplicial identities exactly rather than up to homotopy. These properties led to this construction's independent discovery, its use, and its naming as the simplicial variant in [36].

We start by setting notation for products of spaces and maps, which we will use extensively.

Notation. If S is a finite set, let X^S be the space of all functions from S to X with the product topology. For coordinates we use $(x_s)_{s \in S}$ or just (x_s) when S is understood. Similarly, a product of maps $\prod_{s \in S} f_s$ may be written $(f_s)_{s \in S}$ or just (f_s) . Recall that $\mathbf{n} = \{1, \dots, n\}$.

Definition 4.1. • Let $C_n(\mathbb{R}^m)$ denote the space of $(x_i) \in (\mathbb{R}^m)^{\mathbf{n}}$ such that if $i \neq j$ then $x_i \neq x_j$.

Let $\tilde{C}_n(\mathbb{R}^m)$ be the quotient of $C_n(\mathbb{R}^m)$ by the equivalence relation generated by translating all of the x_i by some v or multiplying them all by the same positive scalar.

- For any $v \in \mathbb{R}^m - 0$, let $u(v) = \frac{v}{\|v\|}$, the unit vector in the direction of v .
- Let $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$ be the closure of the image of $\tilde{C}_n(\mathbb{R}^m)$ under the map (π_{ij}) to $(S^{m-1})^{\binom{\mathbf{n}}{2}}$, where π_{ij} sends the equivalence class of (x_i) to $u(x_i - x_j)$.

Note that (π_{ij}) is not injective – it fails to be so on configurations in which all the x_i lie on some line – so $\tilde{C}_n(\mathbb{R}^m)$ is not a subspace of $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$. But we do have the following theorem, a consequence of Corollary 4.5, Lemma 4.12 and Corollary 5.10 of [37].

Theorem 4.2. *The canonical map $\tilde{C}_n(\mathbb{R}^m) \rightarrow \tilde{C}_n\langle[\mathbb{R}^m]\rangle$ is a homotopy equivalence.*

What makes $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$ manageable is that we can characterize it as a subspace of $(S^{m-1})^{\binom{\mathbf{n}}{2}}$. We extend coordinates for $(u_{ij}) \in (S^{m-1})^{\binom{\mathbf{n}}{2}}$ by letting u_{ji} be $-u_{ij}$ when $j > i$.

Definition 4.3. • A chain, or k -chain, in S is a collection $\{i_1 i_2, i_2 i_3, \dots, i_{k-1} i_k\}$, with all $i_j \in S$ and $i_j \neq i_{j+1}$. Such indices label the edges of a path in the complete graph on S . A chain is a loop, or k -loop, if $i_k = i_1$. A chain is straight if it does not contain any loops. The reversal of a chain is the chain $i_k i_{k-1}, \dots, i_2 i_1$.

- A point $(u_{ij}) \in (S^{m-1})^{\binom{\mathbf{n}}{2}}$ is three-dependent if for any 3-loop L in \mathbf{n} there exist $a_{ij} \geq 0$, with at least one non-zero, such that $\sum_{ij \in L} a_{ij} u_{ij} = 0$.
- If S has cardinality four and is ordered, we may associate to a straight 3-chain C a permutation of S denoted $\sigma(C)$ which relates the order in which indices appear in C to the ordering on S . For example, if $S = \{1, 2, 3, 4\}$, then $\sigma(23, 31, 14) = (2314)$. A complementary 3-chain C^* is a chain, unique up to reversal, which comprises the three pairs of indices not in C .
- A point $(u_{ij}) \in (S^{m-1})^{\binom{\mathbf{n}}{2}}$ is four-consistent if for any $S \subset \mathbf{n}$ of cardinality four and any $v, w \in S^{m-1}$ we have that

$$(2) \quad \sum_{C \in \mathcal{C}^3(S)} (-1)^{|\sigma(C)|} \left(\prod_{ij \in C} u_{ij} \cdot v \right) \left(\prod_{ij \in C^*} u_{ij} \cdot w \right) = 0,$$

where $\mathcal{C}^3 S$ is the set of straight 3-chains in S modulo reversal and $|\sigma(C)|$ is the parity of $\sigma(C)$.

Points in the image of $C_n(\mathbb{R}^m)$ under (π_{ij}) are three-dependent and four-consistent, and also satisfy $u_{ij} = -u_{ji}$, a condition we refer to as anti-symmetry. These properties also hold for $C_n\langle[\mathbb{R}^m]\rangle$, the closure, by continuity. Adding the converse, we have the following, which is Theorem 5.14 of [37].

Theorem 4.4. $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$ is the subspace of all three-dependent, four-consistent points in $(S^{m-1})^{\binom{n}{2}}$.

We will define operad maps on the completions $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$ through coordinates of $(S^{m-1})^{\binom{n}{2}}$. Embed $(S^{m-1})^{\binom{n}{2}}$, and thus $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$, in $(S^{m-1})^{\binom{n}{2}}_+$ as the subspace of $(u_{ij}) \times u_+$ with u_+ equal to the basepoint of S^{m-1} , which we choose to be the south pole $*_S = (0, \dots, 0, -1)$.

Theorem 4.5. The operad structure on $(S^{m-1})^{\mathcal{B}^\bullet}$ restricts to the subspaces $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$.

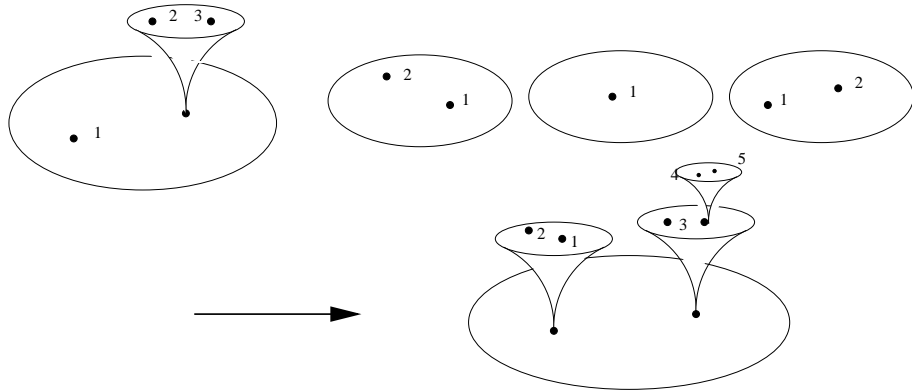
We call the resulting operad, whose n th entry is $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$, the Kontsevich operad \mathcal{K}_m .

Proof. Given a tree T , let $(u_{k\ell}^v)$ be a point in $(S^{m-1})^{\mathcal{B}(T)}$, where v ranges over vertices of T and $k, \ell \in \binom{|v|}{2}$. By Definition 3.1, the operad structure on $(S^{m-1})^{\mathcal{B}^\bullet}$ sends the morphism $T \rightarrow \gamma_n$ to the map given in coordinates by $(u_{k\ell}^v) \mapsto (w_{ij})_{i,j \in \binom{n}{2}}$, where $w_{ij} = u_{J_v(i), J_v(j)}^v$ and v is the join vertex of the leaves i and j .

We verify that if the $(u_{k\ell}^v)$ satisfy three-dependence and four-consistency for each v , then so does (w_{ij}) . For three-dependence, given some w_{ij} , w_{jk} and w_{ki} , there are two cases to consider. In the first case the join in T of leaves i and j lies over that of i and k , so that $w_{jk} = -w_{ki}$ or $0w_{ij} + 1w_{jk} + 1w_{ki} = 0$. In the second case the joins of i and j and k are all equal to the same v , so that the dependence of w_{ij} , w_{jk} and w_{ki} follows from that of $(u_{J_v(i), J_v(j)}^v)$, $(u_{J_v(j), J_v(k)}^v)$ and $(u_{J_v(k), J_v(i)}^v)$. Four-consistency works similarly. Given indices i, j, k and ℓ the pairwise joins could all equal some v , in which case four consistency of these $\{w_{ij}\}$ follows from that of $\{u_{k\ell}^v\}$. Or, if for example the join of i and j lies over those of i, k and ℓ , then $w_{ik} = w_{jk}$ and $w_{i\ell} = w_{j\ell}$, so four-consistency will follow by the canceling of terms which agree but for opposite signs. \square

In [37], we stratify $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$, and in particular the points added in closure. We will not need this stratification explicitly for our applications, but the related geometry is helpful in understanding the operad structure of \mathcal{K}_m . The stratification is indexed by rp-trees with no redundant edges, with the T th stratum being the image of a map from $\prod_{v \in V(T)} \tilde{C}_{|v|}(\mathbb{R}^m)$ to $\tilde{C}_n\langle[\mathbb{R}^m]\rangle$ sending $(x_i^v) \mapsto (u_{ij})$ with $u_{ij} = \pi_{J_v(i), J_v(j)}((x_i)_v)$. Studying this stratification led us to the definition of the choose-two operad. See Section 3 and Theorem 5.14 of [37] for a full development of this geometry, which is illustrated in Figure 4.6.

Figure 4.6. The effect of an operad structure map associated to the morphism $\vee \vee \vee \rightarrow \vee$.



The standard completions $\widetilde{C}_n[\mathbb{R}^m]$ also constitute entries of an operad, which has been more intensively studied [14, 22, 25]. The reason we use \mathcal{K}^m is the following.

Proposition 4.7. *The associative operad maps to the Kontsevich operad, for definiteness by choosing the basepoint $(x_{ij}) \in \widetilde{C}_n\langle[\mathbb{R}^m]\rangle$ with all $x_{ij} = *_S$, for all n .*

Finally, we give a comparison between the little disks operad, which we need to formalize, and the Kontsevich operad.

Definition 4.8. • Recall that the space of n little disks in D^m , the unit disk, denoted $D^m(n)$ is the subspace of $C_n(D^m) \times (0, 1]^n$ of $(x_i) \times (r_i)$ such that the balls $B(x_i, r_i)$ are contained in D^m and have disjoint interiors.

- Let T be a tree whose vertices consist of the root vertex v_0 and a terminal vertex v_e for each root edge e . Thus, $T \rightarrow \gamma_n$, where n is the number of leaves of T , gives rise to one of May's structure maps as in Figure 2.14. Given a label $i \in \mathbf{n}$ let $v(i)$ be the initial vertex for the i th leaf, let $o(i)$ be the label of leaf i within the ordering on edges of $v(i)$ and let $e(i)$ be the label of the root edge for which $v(i)$ is terminal.
- Define $\mathcal{D}^m(T \rightarrow \gamma_n)$ as follows

$$(x_i^v, r_i^v)_{1 \leq i \leq \#v}^{v \in V(T)} \mapsto (y_j, \rho_j)_{j \in \mathbf{n}} \quad \text{where} \quad y_j = x_{e(j)}^{v_0} + r_{e(j)}^{v_0} x_{o(j)}^{v(j)} \quad \text{and} \quad \rho_j = r_{e(j)}^{v_0} r_{o(j)}^{v(j)}.$$

Boardman and Vogt [4] and May [27] showed that connected algebras over \mathcal{D}^m are m -fold loop spaces.

Theorem 4.9. *Let T be a tree with a vertex over each root edge as in Definition 4.8 above. The following diagram commutes up to homotopy,*

$$\begin{array}{ccc} \mathcal{D}^m(T) & \xrightarrow{\mathcal{D}^m(T \rightarrow \gamma_n)} & \mathcal{D}^m(n) \\ p_T \downarrow & & p_n \downarrow \\ \mathcal{K}^m(T) & \xrightarrow{\mathcal{K}^m(T \rightarrow \gamma_n)} & \mathcal{K}^m(n), \end{array}$$

where the vertical maps p_T are the products of projections $p_n : D^m(n) \rightarrow C_n(\mathbb{R}^m)$ composed with the canonical maps $C_n(\mathbb{R}^m) \rightarrow \widetilde{C}_n(\mathbb{R}^m) \rightarrow \widetilde{C}_n\langle[\mathbb{R}^m]\rangle$. Moreover, the vertical maps are homotopy equivalences.

Proof. We define the homotopy explicitly. Define $H : \mathcal{D}^m(T) \times (0, 1] \rightarrow \widetilde{C}_n(\mathbb{R}^m)$ by sending (x_i^v, r_i^v) as in Definition 4.8 and $t \in (0, 1]$ to the equivalence class of $(y_j(t))$ with $y_j(t) = x_{e(j)}^{v_0} + t \cdot r_{e(j)}^{v_0} x_{o(j)}^{v(j)}$. We claim that H extends uniquely to $\overline{H} : \mathcal{D}^m(T) \times [0, 1] \rightarrow \widetilde{C}_n\langle[\mathbb{R}^m]\rangle$, and that \overline{H} coincides with $\mathcal{K}^m(T \rightarrow \gamma_n) \circ p_T$ when $t = 1$. Consider $u_{ij} = \pi_{ij}((y_k(t)))$. If the join of leaves i and j is one of the non-root vertices, so $v(i) = v(j)$, then u_{ij} will be equal to the unit vector in the direction of $x_{o(j)}^{v(j)} - x_{o(i)}^{v(j)}$, independent of t . If the join of leaves i and j is the root vertex, then as t approaches 0, u_{ij} approaches the unit vector in the direction of $x_{e(j)}^{v_0} - x_{e(i)}^{v_0}$. These limiting values coincide with the definition of $\mathcal{K}^m(T \rightarrow \gamma_n)(x_i^v)$.

That the projection $D^m(n) \rightarrow C_n(D^m)$ is a homotopy equivalence is standard, known since the definition of little cubes in [3], so by Theorem 4.2 the maps p_n are homotopy equivalences. \square

In fact, [22] claims that these \mathcal{D}^m and \mathcal{K}^m are homotopy equivalent operads, which we assume to mean that there is a chain of equivalences of maps of operads, that is maps which commute with structure maps exactly. We will not need this stronger claim. Recall that the homology of an operad of spaces with field coefficients is an operad of vector spaces by the Künneth theorem. The homology of the little disks operad has a well-known description.

Definition 4.10. The k th entry of the degree n Poisson operad $\text{Pois}_n(k)$ is the submodule of the symmetric algebra on the free graded Lie algebra over k variables x_1, \dots, x_k spanned by monomials in which

all variables appear exactly once. Monomials are graded by putting all x_i in degree zero and giving the bracket degree n . So for example $[x_1, x_3][x_4, x_2]x_5$ and $x_1x_2 \cdots x_5$ are elements of $\text{Poiss}_3(5)$ of degree nine and zero respectively.

The map $\circ_i : \text{Poiss}_n(j) \otimes \text{Poiss}_n(k) \rightarrow \text{Poiss}_n(j+k-1)$ sends $m_1 \otimes m_2$ to the monomial defined as follows.

- For each j , substitute x_{j+i-1} for x_j in m_2 to obtain $\overline{m_2}$.
- In m_1 , substitute x_{j+i-1} for x_j if $j > i$ and $\overline{m_2}$ for x_i to obtain m .
- Reduce m according to the graded Leibniz rule

$$[a, bc] = [a, b]c + (-1)^{(|a|+n+1)|b|} b[a, c],$$

to obtain an element of $\text{Poiss}_n(i+j-1)$.

The following corollary is essentially a summary of Fred Cohen's famous calculation of the homology of \mathcal{D}^m [10]. We also plan to give an exposition of this result in [38].

Theorem 4.11. *The homology of \mathcal{K}^m is Poiss_{m-1} .*

5. MODELS FOR SPACES OF KNOTS AND IMMERSIONS ARISING FROM THE CALCULUS OF EMBEDDINGS

5.1. A brief overview of the calculus of embeddings. Our main theorems connect the theory of operads to Goodwillie calculus. We first informally introduce some terminology from the calculus of embeddings (see Weiss's [45] for an excellent introduction and [46] for a full treatment), and then precisely state the theorems we use. The main spaces with which we are concerned are related to embeddings and immersions.

Definition 5.1. Let $\text{Emb}(M, N)$ denote the space of embeddings of M in N , topologized as a subspace of the space of C^1 maps. Similarly, let $\text{Imm}(M, N)$ be the space of immersions of M in N . If M and N have boundary, we usually specify some boundary conditions. In particular, if $M = \mathbb{I}$ and $N = \mathbb{I}^m$, we let $*_+ = (0, \dots, 0, 1) \in \mathbb{I}^m$, $*_- = (0, \dots, 0, -1)$ and demand that the endpoints of \mathbb{I} map to $*_+$ and $*_-$ with tangent vectors $*_S$.

By results of Palais [32], these spaces are dominated by simplicial complexes and thus homotopy equivalent to CW-complexes [31].

In the calculus of embeddings, we view spaces of embeddings, immersions and other moduli in differential topology as functors from the poset of open subsets of M to topological spaces, a philosophy originally due to Gromov. Ultimately interested in the value of the functor at the open set which is all of M , we try to use homotopy limits to interpolate that value from values of the functor at simple open sets, namely those which are diffeomorphic to a union of finitely many disjoint open balls. Functors for which interpolation using a finite number of balls works perfectly are called polynomial, and those for which interpolation works in the limit as the number of balls tends to infinity are called analytic. Weiss shows in [46] that polynomial functors are those which satisfy higher-order Mayer-Vietoris conditions, and Goodwillie-Weiss show in [16] that analyticity follows from satisfying those conditions through an increasing range of connectivity. More formally we have the following.

Definition 5.2.

- For any manifold W of dimension m let $\mathcal{U}(M)$ be the category of open subsets of M under inclusion, and let $\mathcal{U}_k(M)$ be the full sub-category of $\mathcal{U}(M)$ of open sets diffeomorphic to $\sqcup_i \mathbb{R}^m$, where $i \leq k$.
- For any contravariant functor F from $\mathcal{U}(M)$ to spaces let $T_k F$ be the functor which sends W to $\varprojlim_{U \in \mathcal{U}_k(W)} F(U)$.
- Let $b_k(F) : T_k F \rightarrow T_{k-1} F$ be the canonical natural transformation defined by restricting $\mathcal{U}_k(M)$ to $\mathcal{U}_{k-1}(M)$, and let $T_\infty F$ be the homotopy inverse limit of the $T_k F$ over these restrictions.

- Let $\eta_k(F) : F \rightarrow T_k F$ be the canonical natural transformation arising from the maps $F(M) \rightarrow F(U)$ for $U \in \mathcal{U}_k(M)$. If by context F is understood, we will use the simpler notation η_k .
- The natural transformations η_k commute with the b_k , so let $\eta_\infty : F \rightarrow T_\infty F$ be the limiting natural transformation.

The sequence $T_0 F \xleftarrow{b_1} T_1 F \xleftarrow{b_2} T_2 F \leftarrow \dots$ is called the Taylor tower for F . Analyticity means that the homotopy inverse limit of this tower is weakly equivalent to F . The motivating example for this circle of ideas is that of immersions.

Theorem 5.3. *If $\dim(M) < \dim(N)$ then for $k \geq 1$, $\eta_k : \text{Imm}(U, N) \rightarrow T_k \text{Imm}(U, N)$ is a weak equivalence for any $U \subseteq M$.*

This theorem follows from Example 2.3 of [46], which says that $\text{Imm}(-, N)$ is a linear functor, and the commentary after Theorem 5.1 of [46]. See also Theorem 5.15 below and Proposition 5.12 of [43]. The embedding functor is not polynomial but by theorems of Goodwillie, Klein and Weiss it is analytic. The following Theorem is essentially Corollary 2.5 of [16].

Theorem 5.4. *If $\dim(M) < \dim(N) - 2$ then $\eta_\infty(\text{Emb})$ is a weak equivalence.*

For $\dim(M) < \dim(N) - 2$ as stated, this theorem requires deep disjunction results of Goodwillie, and surgery results of Goodwillie-Klein [20, 19]. If $\dim(M)$ is less than roughly $\frac{\dim(N)}{2}$, in particular when M is one-dimensional and N has dimension five or greater, there are much easier methods, using only the generalized Blakers-Massey theorem and dimension counting, for proving the needed higher-order Mayer-Vietoris conditions.

5.2. Knot space models through homotopy limits of configuration spaces. Definition 5.2 of $T_k \text{Emb}(M, N)$ seems difficult to manage, being a homotopy limit over a large indexing category. But the building blocks, namely spaces of embeddings of balls, are essentially configuration spaces. Goodwillie, Klein and Weiss have used this observation to give more concrete models for the spaces in this Taylor tower (or for the homotopy fibers of $T_k \rightarrow T_{k-1}$, which are called layers), either as spaces of sections, as in section nine of [46], or as mapping spaces with strongly defined equivariance properties, as in [17]. In the case of knots we have developed three closely-related models for these polynomial approximations [36] and used them for both computational and geometric applications [34, 9]. These models all utilize completions of configuration spaces constructed similarly to the Kontsevich operad.

Definition 5.5. • Let $A_n\langle[\mathbb{I}^m]\rangle$ be the product $(\mathbb{I}^m)^{\mathbf{n}} \times (S^{m-1})^{\binom{n}{2}}$, with coordinates $(x_i) \times (u_{ij})$.
 • Let $C_n\langle[\mathbb{I}^m]\rangle$ be the closure of the image of $C_n(\mathbb{I}^m)$ under $\iota \times (\pi_{ij})$, where ι is the inclusion of $C_n(\mathbb{I}^m)$ in $(\mathbb{I}^m)^{\mathbf{n}}$.
 • Let $C_n\langle[\mathbb{I}^m, \partial]\rangle$ be the closure in $C_{n+2}\langle[\mathbb{I}^m]\rangle$ of the subspace of $C_{n+2}(\mathbb{I}^m)$ with $x_1 = *_+ = (0, \dots, 0, 1)$ and $x_{n+2} = *_-$.

In [37] we study $C_n\langle[\mathbb{I}^m]\rangle$ by relating it to the canonical compactification $C_n[\mathbb{I}^m]$, which is a manifold with corners. We characterize $C_n\langle[\mathbb{I}^m]\rangle$ as a subspace of its defining ambient space, as stated for $\tilde{C}_n\langle[\mathbb{R}^k]\rangle$ in Theorem 4.4. The following is essentially Theorem 5.14 of [37].

Theorem 5.6. *$C_n\langle[I^k]\rangle$ is the subspace of $(x_i) \times (u_{ij})$ such that $(u_{ij}) \in \tilde{C}_n\langle[\mathbb{R}^k]\rangle$ and if $x_i \neq x_j$ then u_{ij} is $u(x_j - x_i)$.*

In our models, we need diagonal maps between configuration spaces. The idea is to add a point “infinitesimally far” from one point in a configuration, but to do so entails choosing a unit tangent vector at that point.

Definition 5.7. Let $C'_n\langle[\mathbb{I}^m]\rangle = C_n\langle[\mathbb{I}^m]\rangle \times (S^{m-1})^{\mathbf{n}}$. Let $A'_n\langle[\mathbb{I}^m]\rangle = (\mathbb{I}^m \times S^{m-1})^{\mathbf{n}} \times (S^{m-1})^{\binom{n}{2}}$, which is canonically diffeomorphic to $(\mathbb{I}^m)^{\mathbf{n}} \times (S^{m-1})^{\frac{n(n+1)}{2}}$. We use coordinates for this latter presentation of the form $(x_i) \times (u_{ij})$ with $i \leq j$, and if $i > j$ we set $u_{ij} = -u_{ji}$.

As in Theorem 4.5 we define maps between the $C'_n\langle[\mathbb{I}^m]\rangle$ at the level of the ambient spaces $A'_n\langle[\mathbb{I}^m]\rangle$, using Theorem 5.6 to check that they restrict appropriately. We are aided by the following combinatorial shorthand.

Definition 5.8. • Given a map of sets $\sigma : R \rightarrow S$ let p_σ^X , or just p_σ , denote the map from X^S to X^R which sends $(x_i)_{i \in S}$ to $(x_{\sigma(j)})_{j \in R}$.
 • Given $\sigma : \mathbf{m} \rightarrow \mathbf{n}$, define $A_\sigma : A'_n\langle[\mathbb{I}^m]\rangle \rightarrow A'_m\langle[\mathbb{I}^m]\rangle$ as $p_\sigma^{\mathbb{I}^m} \times p_{\sigma(2)}^{S^{k-1}}$, where $\sigma(2) = \sigma \times \sigma|_{\binom{n}{2}}$.

Proposition 5.9 (Proposition 6.6 of [37]). *The restriction A_σ to $C'_n\langle[\mathbb{I}^m]\rangle$ maps to $C'_m\langle[\mathbb{I}^m]\rangle$. If σ sends $1 \rightarrow 1$ and $n \rightarrow m$ then A_σ also restricts to a map, which we call F_σ , from $C'_{n-2}\langle[\mathbb{I}, \partial]\rangle$ to $C'_{m-2}\langle[\mathbb{I}, \partial]\rangle$.*

We may now define diagonal maps on compactified configuration spaces with tangential data.

Definition 5.10. Let $\delta^i : C'_n\langle[\mathbb{I}^m, \partial]\rangle \rightarrow C'_{n+1}\langle[\mathbb{I}^m, \partial]\rangle$ be F_{σ_i} where $\sigma_i : \underline{n+3} \rightarrow \underline{n+2}$ sends j to itself if $j \leq i$ or $j-1$ if $j > i$.

A final key property of this compactification is that it is functorial for embeddings. The proof of the following theorem is identical to that of Corollary 4.8 of [37], using Theorem 5.8 of [37] and the fact that $C_n\langle[\mathbb{I}]\rangle = \Delta^n$. Recall that for a nonzero vector $v \in \mathbb{R}^m$, $u(v) = \frac{v}{\|v\|}$.

Theorem 5.11. *For an embedding $f : \mathbb{I} \rightarrow \mathbb{I}^k$ there is an evaluation map $ev_n(f) : \Delta^n \rightarrow C_n\langle[\mathbb{I}^k]\rangle$ which extends the map from the interior of Δ^n to $C'_n(\mathbb{I}^k)$ sending (t_i) to $(f(t_i)) \times (u(f'(t_i)))$.*

One of the main themes of [36] and of [43] is connecting this evaluation map with the calculus of embeddings. Applying this calculus to embeddings of the unit interval is simpler than to embeddings of higher-dimensional manifolds because the category $\mathcal{U}_k(\mathbb{I})$ may be replaced by the category of subsets of a finite set (see Definition 2.7).

Definition 5.12. Let \mathcal{D}_k^m be the functor from $P_0(k)$ to spaces which sends $S \subseteq [k]$ to $C'_{\#S-1}\langle[\mathbb{I}^m, \partial]\rangle$ and which sends the inclusion $S \subset S \cup j$ to the map δ^i where i is the number of elements of S less than j . Let $D_k^m = \varprojlim \mathcal{D}_k^m$.

In the notation of [36], \mathcal{D}_k^m would be $\mathcal{D}_k\langle[\mathbb{I}^m]\rangle$. Because the realization of $P_0(k)$ is Δ^k and all of the maps δ^i are inclusions of subspaces, D_k^m is a subspace of $\text{Maps}(\Delta^k, C'_k\langle[\mathbb{I}^m, \partial]\rangle)$. If $f \in \text{Emb}(\mathbb{I}, \mathbb{I}^m)$ is a knot, $ev_k(f)$ defines an element of D_k^m , as we may simply check that if $t_j = t_{j+1}$ for some point $(t_i) \in \Delta^k$ then the image of $ev_k(f)((t_i))$ is in the image of δ^j . By abuse, let ev_k denote the adjoint map from $\text{Emb}(\mathbb{I}, \mathbb{I}^m)$ to D_k^m . Building on the simpler “cutting method” definition of $T_k\text{Emb}(\mathbb{I}, \mathbb{I}^m)$, as described in Section 3 of [36], Lemma 5.18 and the proof of Theorem 5.3 of [36] establish the following.

Theorem 5.13. *D_k^m is homotopy equivalent to $T_k\text{Emb}(\mathbb{I}, \mathbb{I}^m)$, and ev_k agrees with η_k in the homotopy category.*

We next take a similar point of view for immersions of an interval, in order to arrive at our model for E_m . This point of view is taken in Section 5.3 of [43], which also considers the homotopy fiber of the map from knots to immersions. Indeed, our Theorem 5.15 and Proposition 5.16 below overlap significantly with Propositions 5.12 and 5.13 of [43].

Definition 5.14. • Let $d^i : (S^{m-1})^j \rightarrow (S^{m-1})^{j+1}$ be the i th diagonal inclusion, which on coordinates repeats the i th entry. By convention, for $i = 0$ and $i = j+1$ we insert the basepoint $*_S$ as the first, respectively last, coordinate.
 • Let \mathcal{G}_k^m be the functor from $P_0(k)$ to spaces which sends $S \subseteq [k]$ to $(S^{m-1})^{\#S-1}$ and which sends the inclusion $S \subset S \cup j$ to the diagonal map d^i where i is the number of elements of S less than j .
 • Let $G_k^m = \varprojlim \mathcal{G}_k^m$.

As was true for D_k^m , G_k^m is a subspace of the space of maps from Δ^k to the terminal space of \mathcal{G}_k^m , namely $(S^{m-1})^k$. The evaluation map for immersions is the unit derivative map. By abuse, let $ev_k : \text{Imm}(\mathbb{I}, \mathbb{I}^m) \rightarrow G_k^m$ send f to the map which sends t_1, \dots, t_k to $uf'(t_1), \dots, uf'(t_k)$.

Theorem 5.15. *If $k \geq 1$, G_k^m is homotopy equivalent to $T_k \text{Imm}(\mathbb{I}, \mathbb{I}^m)$, and thus to $\text{Imm}(\mathbb{I}, \mathbb{I}^m)$. Moreover, ev_k agrees with η_k in the homotopy category.*

Sketch of proof. There are many ways to establish this theorem. By the Hirsch-Smale theorem [39], $\text{Imm}(\mathbb{I}, \mathbb{I}^m)$ is homotopy equivalent to ΩS^{m-1} , through the unit derivative map. But $ev_1 : \text{Imm}(\mathbb{I}, \mathbb{I}^m) \rightarrow \text{holim}(* \rightarrow S^{m-1} \leftarrow *)$ is also the unit derivative map, which establishes the theorem for $k = 1$. For the other k , we may use Lemma 2.9, since \mathcal{G}_k^m is the pull-back of the standard cosimplicial model for ΩS^{m-1} through the functor c_k of Definition 2.8. The k th totalization of this cosimplicial model, which is fibrant, is homeomorphic to ΩS^{m-1} if $k \geq 1$, from which it follows that \mathcal{G}_k^m is homotopy equivalent to ΩS^{m-1} . The map from ΩS^{m-1} to the k th totalization, and thus G_k^m , is through evaluation of the unit derivative. \square

Let $\tau : \text{Emb}(\mathbb{I}, \mathbb{I}^m) \rightarrow \text{Imm}(\mathbb{I}, \mathbb{I}^m)$ denote the inclusion. Let $\rho_k^m : \mathcal{D}_k^m \rightarrow \mathcal{G}_k^m$ denote the map of diagrams defined on each entry by projection from $C'_n \langle [\mathbb{I}^m, \partial] \rangle = C_n \langle [\mathbb{I}^m, \partial] \rangle \times (S^{m-1})^n$ onto $(S^{m-1})^n$, and let p_k^m also denote the induced map on homotopy limits.

Proposition 5.16. *The square*

$$\begin{array}{ccc} \text{Emb}(\mathbb{I}, \mathbb{I}^m) & \xrightarrow{\tau} & \text{Imm}(\mathbb{I}, \mathbb{I}^m) \\ ev_k \downarrow & & \downarrow ev_k \\ D_k^m & \xrightarrow{p_k^m} & G_k^m. \end{array}$$

commutes. Moreover, p_k^m agrees with $T_k(\tau)$ in the homotopy category.

Sketch of proof. The commutativity of the diagram is immediate from the definitions. That p_k^m agrees with $T_k(\tau)$ in the homotopy category ultimately follows from the fact that for U a disjoint union of $k+2$ open intervals, two of which contain endpoints of \mathbb{I} and thus are fixed at one end, we have $\text{Emb}(U, \mathbb{I}^m) \simeq C'_k \langle [\mathbb{I}^m, \partial] \rangle$, $\text{Imm}(U, \mathbb{I}^m) \simeq (S^{m-1})^k$ and the inclusions from embeddings to immersions coincides with projection, as in the definition of p_k^m . \square

5.3. A closer look at E_m .

Proposition 5.17. *The inclusion $\tau : \text{Emb}(\mathbb{I}, \mathbb{I}^m) \rightarrow \text{Imm}(\mathbb{I}, \mathbb{I}^m)$ is null-homotopic, so*

$$E_m \simeq \text{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega \text{Imm}(\mathbb{I}, \mathbb{I}^m) \simeq \text{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega^2 S^{m-1}.$$

Proof. Given $f \in \text{Emb}(\mathbb{I}, \mathbb{I}^m)$ consider the map $\rho(f) : \Delta^2 \rightarrow S^{m-1}$ which sends t_1, t_2 to either $u(f(t_2) - f(t_1))$ if $t_1 \neq t_2$ or $u(f'(t))$ if $t_1 = t_2 = t$. We may view $\rho(f)$ as a homotopy between $ev_1(f)$, which is the restriction to the $t_1 = t_2$ edge, and the restriction to the $t_1 = 0$ and $t_2 = 1$ edges. But the restriction to these latter two edges is canonically null-homotopic, since their images lie in the southern hemisphere of S^{m-1} . Thus, ev_1 restricted to $\text{Emb}(\mathbb{I}, \mathbb{I}^m)$ is null-homotopic. Since ev_1 is an equivalence on $\text{Imm}(\mathbb{I}, \mathbb{I}^m)$ this implies that the inclusion of $\text{Emb}(\mathbb{I}, \mathbb{I}^m)$ is null-homotopic.

That $E_m \simeq \text{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega \text{Imm}(\mathbb{I}, \mathbb{I}^m)$ is immediate from its definition as the homotopy fiber of this inclusion, and that this is in turn weakly equivalent to $\text{Emb}(\mathbb{I}, \mathbb{I}^m) \times \Omega^2 S^{m-1}$ follows from the Hirsch-Smale theorem [39]. \square

The elements of E_3 , which by the above is homotopy equivalent to $\text{Emb}(\mathbb{I}, \mathbb{I}^3) \times \Omega^2 S^2$, are naturally equipped with framings.

Proposition 5.18 (Proposition 5.14 from [43]). *The components of E_3 are canonically identified with isotopy classes of framed knots whose framing is even.*

Given an element of E_3 , a knot with a homotopy through immersions to the unknot, Volic “carries” the zero framing on the unknot through the homotopy to define a framing number on the knot, which is necessarily even. Note that E_m is also the homotopy fiber of the inclusion of the space of framed embeddings in the space of framed immersions. When $m = 3$ the space of framed immersions is homotopy equivalent to $\Omega SO(3)$, so the long-exact sequence in homotopy for this fibration reads

$$\cdots \pi_2(SO(3)) = 0 \rightarrow \pi_0(E_3) \rightarrow \pi_0 \text{Emb}(\mathbb{I}, \mathbb{I}^3) \times \mathbb{Z} \rightarrow \pi_1(SO(3)) \cong \mathbb{Z}/2 \rightarrow 0,$$

consistent with the calculation that $\pi_0(F_3) \cong \pi_0(\text{Emb}(\mathbb{I}, \mathbb{I}^3) \times 2\mathbb{Z})$.

6. THE MAIN RESULT

We assemble our work to this point to prove the main result. As needed for the calculus of functors, extend E_m to be a functor on the open sets of \mathbb{I} by sending U to the homotopy fiber of the inclusion $\text{Emb}(U, \mathbb{I}^m) \rightarrow \text{Imm}(U, \mathbb{I}^m)$. We will recover models for E_m from those for embeddings and immersion spaces.

Lemma 6.1. *If A and B are two functors from $\mathcal{U}(M)$ to spaces with a natural transformation τ between them, and F is defined by $F(U) = \text{hofib}(\tau : A(U) \rightarrow B(U))$, then $T_k(F) = \text{hofib}(T_k(A) \rightarrow T_k(B))$.*

Proof. The equality is immediate from the definition of T_k , since taking homotopy fibers commutes with taking homotopy limits. \square

In defining a fiber to ρ_k^m we are led to the following.

Definition 6.2. Let $e_i : A_n\langle[\mathbb{I}^m]\rangle \rightarrow A_{n+1}\langle[\mathbb{I}^m]\rangle$ send $(u_{j\ell})$ to $(v_{j\ell})$ where $v_{i,i+1} = *_{\mathcal{S}}$, the basepoint of S^{m-1} and other $v_{j\ell}$ are equal to $u_{\sigma_i(j)\sigma_i(\ell)}$. As before $\sigma_i(j) = j$ or $j - 1$ if $j < i$ or $j > i$ respectively. By abuse, use e_i to denote its restriction to $C_n\langle[\mathbb{I}^m, \partial]\rangle$ mapping to $C_{n+1}\langle[\mathbb{I}^m, \partial]\rangle$, as one can check using Theorem 5.6.

Alternately, $e_i : C_n\langle[\mathbb{I}^m, \partial]\rangle \rightarrow C_{n+1}\langle[\mathbb{I}^m, \partial]\rangle$ is the restriction of δ^i to $C_n\langle[\mathbb{I}^m, \partial]\rangle \times (*_{\mathcal{S}})^n \subset C'_n\langle[\mathbb{I}^m, \partial]\rangle$.

Definition 6.3. Let \mathcal{F}_k^m be the functor from $P_0(k)$ to spaces which sends $S \subseteq [k]$ to $C_{\#S-1}\langle[\mathbb{I}^m, \partial]\rangle$ and which sends the inclusion $S \subset S \cup j$ to the map e^j where i is the number of elements of S less than j . Let $F_k^m = \varprojlim \mathcal{F}_k^m$.

Theorem 6.4. F_k^m is homotopy equivalent to $T_k E_m$. For $m > 3$, $\eta_\infty : E_m \rightarrow \varprojlim T_k E_m$ is a weak equivalence.

Proof. We use the models D_k^m and G_k^m for $T_k \text{Emb}$ and $T_k \text{Imm}$ as given in Theorems 5.13 and 5.15 respectively. By Proposition 5.16, $p_k^m : D_k^m \rightarrow G_k^m$ agrees with T_k of the inclusion from embeddings to immersions. Applying Lemma 6.1 with $A = \text{Emb}(-, \mathbb{I}^m)$, $B = \text{Imm}(-, \mathbb{I}^m)$, and the natural transformation between them be the standard inclusion, we have that $T_k E_m = \text{hofib } p_k^m$.

If a map of diagrams indexed by $P_0(k)$ is a fibration object-wise, then the induced map on homotopy limits is a fibration and the fiber is given by the homotopy limit of the fibers object-wise (see for example Lemma 3.5 of [9]). Because ρ_k^m is a fibration object-wise, we identify $\text{hofib } p_k^m = \text{hofib}(\varprojlim \rho_k^m)$ with such a homotopy limit of object-wise fibers. By our definition, the diagram of fibers is \mathcal{F}_k^m , whose homotopy limit is F_k^m , establishing the first half of the theorem.

The second half of the theorem is immediate from Theorems 5.3 and 5.4. \square

Because of Proposition 5.17, we could alternately extend E_m to a functor on $\mathcal{U}(\mathbb{I})$ by setting $E_m^!(U) = \text{Emb}(U, \mathbb{I}^m) \times \Omega \text{Imm}(U, \mathbb{I}^m)$. The extension $E_m^!$ would lead to a set of approximations to E_m different from the F_k^m .

Recall Proposition 4.7 that \mathcal{K}_m is an operad with multiplication, which using Definition 2.17 has an associated cosimplicial object. We translate from F_k^m to $\widetilde{\text{Tot}}(\mathcal{K}_m^\bullet)$, essentially through the standard projection $C_n\langle[\mathbb{I}^m, \partial]\rangle \rightarrow \widetilde{C}_n\langle[\mathbb{R}^m]\rangle$. We modify both $C_n\langle[\mathbb{I}^m, \partial]\rangle$ and this projection to define a natural transformation.

Definition 6.5. • Let $\varepsilon \leq \frac{1}{6}$. For $x \in \mathbb{R}^m$, let $d_+(x)$ be the distance in \mathbb{R}^m from x to $*_+ = (0, \dots, 0, 1)$ and $d_-(x)$ be the distance to $*_-$. Let $\gamma_j : \mathbb{R}^m \rightarrow \mathbb{R}$ be projection onto the j th coordinate.

- Let $C_n\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ be the subspace of $(x_i) \times (u_{ij}) \in C_n\langle[\mathbb{I}^m, \partial]\rangle$ where if $d_+(x_i)$ and $d_+(x_j)$ are less than ε and $i < j$ then $\gamma_k(x_i) = \gamma_k(x_j)$ for $k < m$ and $\gamma_m(x_i) \geq \gamma_m(x_j)$. Moreover, if $x_i = x_j$ and $i < j$ then $u_{ij} = *_{\mathcal{S}}$.
- Let $\mathcal{F}_{k,\varepsilon}^m$ be the functor from $P_0(k)$ to spaces which sends $S \subseteq [k]$ to $C_{\#S-1}\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ and which sends the inclusion $S \subset S \cup j$ to the map e^i where i is the number of elements of S less than j .
- Let $F_{k,\varepsilon}^m = \varprojlim \mathcal{F}_{k,\varepsilon}^m$.

Proposition 6.6. *The map $F_{k,\varepsilon}^m \rightarrow F_k^m$, induced by the natural transformation $\iota : \mathcal{F}_{k,\varepsilon}^m \rightarrow \mathcal{F}_k^m$ which at each entry is the canonical inclusion, is a homotopy equivalence.*

Proof. It suffices to check ι is a homotopy equivalence object-wise, for which we adapt the machinery developed in [37] for compactified configuration spaces. Both $C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ and $C_k\langle[\mathbb{I}^m, \partial]\rangle$ are quotients of the canonical Axelrod-Singer compactifications which we call $C_k[\mathbb{I}^m, \partial_\varepsilon]$ and $C_k[\mathbb{I}^m, \partial]$ respectively; see Definitions 1.3 and 4.18 of [37] for the definition of $C_k[\mathbb{I}^m, \partial]$, which can be modified as in Definition 6.5 for $C_k[\mathbb{I}^m, \partial_\varepsilon]$. These quotient maps are homotopy equivalences, by the proof of Theorem 5.10 of [37], which applies verbatim in these cases.

$C_k[\mathbb{I}^m, \partial_\varepsilon]$ retracts to its subspace $C_k[\mathbb{I}^m - N_\pm^\varepsilon]$, where N_\pm^ε is the union of the ε neighborhoods of $*_+$ and $*_-$ by scaling the x_i by $1 - \varepsilon$. Both $C_k[\mathbb{I}^m, \partial]$ and $C_k[\mathbb{I}^m - N_\pm^\varepsilon]$ are manifolds with corners (see Theorem 4.4 of [37]), and thus are homotopy equivalent to their interiors, $C_k(\text{Int}(\mathbb{I}^m))$ and $C_k(\text{Int}(\mathbb{I}^m - N_\pm^\varepsilon))$ respectively. But these interior configuration spaces are diffeomorphic, since $\text{Int}(\mathbb{I}^m)$ and $\text{Int}(\mathbb{I}^m - N_\pm^\varepsilon)$ are. Composing this diffeomorphism with the previous homotopy equivalences establishes the equivalence of $C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ and $C_k\langle[\mathbb{I}^m, \partial]\rangle$ and thus establishes the result. \square

We use $C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle$ because they readily project to $\widetilde{C}_k\langle[\mathbb{R}^m]\rangle$ in a way compatible with its cosimplicial structure maps.

Definition 6.7. • Let $(a_i)_{i=1}^m$, $a_i \in \mathbb{R}$ denote a point in \mathbb{R}^m . Define $\lambda_+ : (\mathbb{R}^m - *_+) \rightarrow \mathbb{R}^m$ by sending (a_i) to (b_i) where if $i \neq m$ then $b_i = a_i$ and

$$b_m = \begin{cases} \frac{\varepsilon a_m}{d_+(a_i)} & d_+(a_i) < \varepsilon \\ a_m & d_+(a_i) \geq \varepsilon. \end{cases}$$

Define $\lambda_- : (\mathbb{R}^m - *_-) \rightarrow \mathbb{R}^m$ similarly, and let $\lambda = \lambda_+ \circ \lambda_-$.

- Define $\pi_k : C_k\langle[\mathbb{I}^m, \partial_\varepsilon]\rangle \rightarrow \widetilde{C}_k\langle[\mathbb{R}^m]\rangle \subset (S^{m-1})^{\binom{k}{2}}$ by sending $(x_i) \times (u_{ij})$ to (v_{ij}) where v_{ij} is:
 - $u(\lambda(x_i) - \lambda(x_j))$ if $x_i \neq x_j$ and neither equals $*_+$ or $*_-$.
 - The Jacobian on λ applied to u_{ij} if $x_i = x_j$.
 - $*_{\mathcal{S}}$, if $x_i \neq x_j$ and either $x_i = *_+$ or $x_j = *_-$.

Proposition 6.8. π_k is continuous.

Proof. We first identify π_k on the subspace $t C_k[\mathbb{I}^m - (*_+ \cup *_-)]$ with the composite of $C_k\langle[\lambda]\rangle$, the map on configuration spaces induced by the embedding λ (see Corollary 4.8 of [37]), and the canonical projection $C_k\langle[\mathbb{R}^m]\rangle$ to $\widetilde{C}_k\langle[\mathbb{R}^m]\rangle$. What remains is to check continuity on the subspace in which some

$x_n = *_+$. Consider a sequence $\{(x_i^\ell), (u_{ij}^\ell)\}_{\ell=1}^\infty$ with limit point $(x_i^\infty) \times (u_{ij}^\infty)$, so that $x_n^\infty = *_+$. We show that its image under π_k has v_{nj} which approaches $*_S$ if $x_j^\infty \neq *_+$ or $n < j$ or which approaches $-*_S$ otherwise. For each j , either $x_j^\infty \in N_+^\varepsilon$, which is also true for ℓ sufficiently large, in which case u_{nj} must be $*_S$ if $n < j$ or $-*_S$ if $n > j$, so that the sequence v_{nj} would be eventually constant at $*_S$ or $-*_S$. Or if $x_j^\infty \notin N^\varepsilon$ then as $x_n^\ell \mapsto *_+$ the last coordinate of $\lambda_+(x_i^\ell)$ becomes arbitrarily large. Because $x_j^\ell \mapsto x_j$ stays in \mathbb{I}^m we have $u(\lambda(x_n^\ell) - \lambda(x_j^\ell)) \mapsto *_S$. Continuity when some $x_n = *_-$ works similarly. \square

We now may assemble our main result, Theorem 1.1, which casts the embedding calculus tower for E_m in the language of operads. For convenience, we restate the theorem here.

Theorem 6.9. *The k th approximation to E_m in the embedding calculus, namely $T_k E_m$, is weakly equivalent to $\widetilde{\text{Tot}}^k \mathcal{K}_m^\bullet$.*

Proof. We will check that the maps π_k assemble to a natural transformation of functors from \mathcal{F}_k^m to $\mathcal{K}_m^\bullet \circ c_k$, with c_k as in Definition 2.8, which gives rise to a weak equivalence on homotopy limits. Theorem 6.4 then says that the homotopy limit of \mathcal{F}_k^m is weakly equivalent to $T_k E_m$, and Lemma 2.9 implies that the homotopy limit of $\mathcal{K}_m^\bullet \circ c_k$ is weakly equivalent to $\widetilde{\text{Tot}}^k \mathcal{K}_m^\bullet$, establishing the theorem.

For the assembled π_k to be a natural transformation, we must have $\pi_k \circ e^i = d^i \circ \pi_k$. For most i this is immediate to check, as repeating coordinates and passing to the quotient $\widetilde{C}_k \langle [\mathbb{R}^m] \rangle$ are processes which clearly commute. The $i = 0$ and $i = k + 1$ cases require the modifications we made in Definition 6.7. For $\mathcal{K}_m^\bullet \circ c_k$ we trace through Definitions 2.17 and 3.1, Theorem 4.5 and Proposition 4.7 to see that d^{k+1} takes a point $(u_{ij}) \in \widetilde{C}_k \langle [\mathbb{R}^m] \rangle$, leaves all these u_{ij} unchanged, and adds $u_{i,k+1} = *_S$ for all i to obtain a point in $\widetilde{C}_{k+1} \langle [\mathbb{R}^m] \rangle$. On the other hand, e^{k+1} adds the $k + 1$ st point to the configuration at $*_-$, which under π_k will also lead to all $u_{i,k+1} = *_S$. The $i = 0$ case works similarly.

The fact that the assembled π_k induce a weak equivalence on homotopy limits follows from it being a homotopy equivalence object-wise. We already know from the proof of Proposition 6.6 that $C_k \langle [\mathbb{I}^m, \partial_\varepsilon] \rangle$ is homotopy equivalent to the subspace $C_k(\text{Int}(\mathbb{I}^m - N_\pm^\varepsilon))$, which is diffeomorphic to $C_k(\mathbb{R}^m)$. Composed with this diffeomorphism on this subspace, π_k is the standard projection $C_k(\mathbb{R}^m) \rightarrow \widetilde{C}_k(\mathbb{R}^m)$ followed by the canonical map to $\widetilde{C}_k \langle [\mathbb{R}^m] \rangle$ which is a homotopy equivalence by Corollaries 4.5 and 5.9 of [37]. \square

The first half of Corollary 1.2, which states that for $m > 3$ the totalization of the Kontsevich operad faithfully models the weak homotopy type of E_m , now follows from Theorem 6.4. The second half of Corollary 1.2, which states that for $m = 3$ all real-valued finite-type framed knot invariants are pulled back from this operad model, follows from the main results of [43]. We may take any framed knot, double its framing, and then apply Proposition 5.18 to get a corresponding component of E_3 . In [43], Volic uses Bott-Taubes integrals to define finite-type invariants on the Taylor tower for E_3 , which we have now identified with the Tot tower for \mathcal{K}_3 . See in particular Theorems 1.2 and 1.3 and Section 6.4 of [43].

7. OBSERVATIONS AND CONSEQUENCES

7.1. Spectral sequences. The results in this section parallel those of section 7 of [36]. Applying the homotopy spectral sequence of Proposition 2.10 for \mathcal{K}_m^\bullet we immediately have the following.

Theorem 7.1. *There is a spectral sequence converging to $\pi_*(\widetilde{\text{Tot}} \mathcal{K}_m^\bullet)$ with*

$$E_1^{-p,q} = \bigcap \ker s^k_* \subseteq \pi_q(C_p(\mathbb{R}^m)).$$

The d_1 differential is the restriction to this kernel of the map

$$\Sigma_{i=0}^{p+1} (-1)^i d^i_* : \pi_q(C_{p-1}(\mathbb{R}^m)) \rightarrow \pi_q(C_p(\mathbb{R}^m)).$$

Theorem 6.4 implies that this spectral sequence computes homotopy groups of E_m when $m \geq 4$. Except for in the $p = 1$ column, this spectral sequence coincides exactly with that studied with rational coefficients in [34], so we do not give a more explicit description here. Kontsevich [23] has also examined the rows of this spectral sequence.

For $m = 3$, the case of classical knots, we conjecture that $\eta_k : E_m \rightarrow T_k E_m$ is a universal type- $(k - 1)$ framed knot invariant over the integers. For $k \leq 3$, we may deduce this from the main results of [9]. In unpublished work, Conant has shown that the entries $E_{-k,k}^2$ of this spectral sequence are isomorphic to the module of primitive weight systems of degree $k - 1$ over the integers, a first step to this conjecture in full generality.

In light of Theorem 4.11, the homology spectral sequence from Theorem 2.11 has a pleasant description. Recall Definition 2.17, which for operads of vector spaces introduces the notation of $HH^*(\mathcal{O})$ for the total cohomology of the associated cosimplicial object.

Theorem 7.2. *There is a spectral sequence with $E_{-p,q}^2 = HH^{p,q}(\text{Pois}_m)$ which for $m \geq 4$ converges to the homology of $\widetilde{\text{Tot}}\mathcal{K}_m^\bullet$, and thus of E_m .*

Proof. If we use the second description of the homology spectral sequence from Theorem 2.11, then $E_{-*,*}^1$ will be $H_*(\mathcal{K}_m^\bullet)$, which is the Poisson operad by Theorem 4.11. The induced operad with multiplication structure on the Poisson operad is the standard one. Thus, the d^1 differential will coincide with the differential for total cohomology of the Poisson operad, and the E^2 term will be the total (or Hochschild) cohomology of the Poisson operad as stated.

It remains to check the convergence conditions of Theorem 2.11. In the case of the Kontsevich operad, the entries $\mathcal{K}_m^k = \widetilde{C}_k\langle[\mathbb{R}^m]\rangle$ are homotopy equivalent to $C_k(\mathbb{R}^m)$, which are simply connected if $m \geq 3$. Using the first definition of Theorem 2.11, we start with $H_*(C_p(\mathbb{R}^m))$ and explicitly understand the kernels of the maps s_*^i . We use Theorem 4.11 and Definition 4.10 to identify $H_*(C_p(\mathbb{R}^m))$ in terms of products of brackets in variables x_1, \dots, x_p . Tracing through the definitions of the associated cosimplicial object, s^i sends a product of brackets in the x_j to either zero, if the variable x_i appears in a bracket, or the monomial obtained by removing x_i and relabeling x_j to x_{j-1} for $j > 1$, if x_i does not appear in a bracket. Therefore to be in the kernel of all of the s^i , all of the variables x_i must appear in a bracket, so there must be at least $\frac{k}{2}$ brackets, leading to a total degree of at least $\frac{k(m-1)}{2}$. For $m > 3$, this is greater than k and thus gives the estimate needed for application of Theorem 2.11. \square

This spectral sequence in real-valued cohomology coincides with the homotopy spectral sequence of the Taylor tower for the functor to spectra $U \mapsto \mathbb{R} \wedge E_m(U)$, the real Eilenberg-MacLane spectrum smashed with $E_m(U)$. For $m = 3$, Volic's results [43, 44] imply that the map from the knot space to this Taylor tower serves as a universal framed finite-type invariant over the real numbers.

7.2. A little two-cubes action from the McClure-Smith framework. Theorem 1.1 fits perfectly into the framework created by McClure and Smith in their solution of the Deligne conjecture over the integers [28]. One of their central results is the following.

Theorem 7.3. *The totalization of the associated cosimplicial object of an operad with multiplication admits an action of an operad equivalent to the little 2-cubes operad, as does its homotopy invariant totalization.*

Proof. We are simply collecting results from [28] and [29]. For the standard totalization, we are simply quoting Theorem 3.3 in [28]. For the homotopy invariant totalization, Theorem 15.3 of [29] says that Tot of any cosimplicial space with what is called a Ξ^2 -structure has an action of an operad equivalent to the little 2-cubes. Proposition 10.3 of [29] identifies an operad with multiplication structure on a sequence of spaces with a Ξ^2 structure. \square

Example 7.4. *Consider the cosimplicial model for the space of maps from S^2 to X , namely $X^{S^2_\bullet}$. By Theorem 3.3 $S^2_\bullet \cong \mathcal{B}^\bullet$, so there is an operad structure on this collection of spaces. In order to get an*

operad with multiplication, we restrict each $X^{\mathcal{B}^n} = (x_\alpha)$ to the subspace in which $x_+ = *$, where $+$ is the basepoint of \mathcal{B}^n and $*$ is the base point of X . The operad structure maps restrict appropriately, and we obtain $X_\star^{S^2}$, to which the associative operad maps at each level to the point with all $x_\alpha = *$.

Applying Theorem 7.3, the totalization of $X_\star^{S^2}$ is a little 2-cubes space, and we know that its totalization is $\Omega^2 X$. McClure and Smith fully develop this example (more generally for the standard model of $\Omega^n X$) in Section 11 of [29]. They show that the little 2- (or n -) cubes action which arises in these examples coincide with the standard ones.

We can immediately establish Theorem 1.4, one of our main results, which we restate here.

Theorem 7.5. *For any m , there is a little two-cubes action on $\widetilde{\text{Tot}}(\mathcal{K}_m^\bullet)$. For $m > 3$, E_m is a two-fold loop space.*

Proof. Applying Theorem 7.3 for the Kontsevich operad with its given multiplication establishes the two-cubes action.

By Theorem 1.1, if $m \geq 4$, E_m is homotopy equivalent to $\widetilde{\text{Tot}}(\mathcal{K}_m^\bullet)$, so it has a 2-cubes action as well. But E_m is connected for $m \geq 4$, since it is the product of $\Omega(\text{Imm}(\mathbb{I}, \mathbb{I}^m)) \simeq \Omega^2 S^{m-1}$ and $\text{Emb}(\mathbb{I}, \mathbb{I}^m)$ which are both connected (that the latter space is connected is because any path through maps from an embedding to the standard one becomes an isotopy once put in general position). By the recognition theorem of [3, 4], E_m is a 2-fold loop space. \square

Our operad model for E_m has already been important for closer examination of the homotopy type of E_m . In [24], Volic and Lambrechts establishes a formality result for this model, which determines the rational homotopy type of E_m for $m > 3$.

A two-cubes action has been used with spectacular success in dimension three. In [7], Budney constructs a little two-cubes action directly on the space of framed (long) knots in any dimension. He goes on to show that the two-cubes action is free when $m = 3$, generated by the components of prime knots. In [8], he determines the homotopy types of these prime components in terms of the JSJ decompositions of prime knots.

In further work, we may develop a model for framed knots closely related to those studied in this paper. Namely, there is an operad whose entries are $\widetilde{C}_n(\mathbb{R}^m) \times (SO(m))^n$, where in the operad composition, the elements of $SO(m)$ act on the configurations (compare with the operad of “turning balls”). The techniques in this paper should also show that the totalization of this operad models spaces of framed knots. It may then be fruitful to compare Budney’s geometric two-cubes action to the one which arises from the McClure-Smith machinery, as well as to combinatorial product and bracket structures on homology defined by Tourtchine [40].

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